Importance of Anisotropy on Buckling of Compression-Loaded Symmetric Composite Plates

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The differential equation governing buckling of symmetrically laminated composite plates loaded in compression is presented in nondimensional form. From this equation, nondimensional material coefficients are obtained, and a nondimensional parameter is presented that is used to assess when anisotropic bending stiffnesses can be neglected in a buckling analysis. Results obtained using finite element analyses are presented that show how boundary conditions, aspect ratio, fiber orientation, stacking sequence, and thickness affect the importance of the anisotropic bending stiffnesses.

Nomenclature

a,b	= dimensions of rectangular plate parallel to x and y axes, respectively
m	= stacking sequence number
$\mathcal{O}$	= symbol denoting order of
w	= out-of-plane displacement of the plate
$D_{11},D_{12},D_{22},D_{66}$	= orthotropic plate bending stiffnesses
D_{16},D_{26}	= anisotropic plate bending stiffnesses
K	= buckling coefficient defined by Eq. (4)
K^*	= buckling coefficient neglecting D_{16} and D_{26} terms
N_x,N_y,N_{xy}	= membrane stress resultants
α	= nondimensional wavelength parameter defined by Eq. (5)
β	= nondimensional twisting stiffness parameter defined by Eq. (6)
γ,δ	= nondimensional anisotropic coefficients defined by Eq. (7)
η,ξ	= nondimensional coordinates defined by Eq. (3)
λ	= buckling mode half-wavelength shown in Fig. 1
μ	= anisotropic parameter defined by Eq. (16)

Introduction

AN important consideration in the analysis of composite structures is the effect of neglecting anisotropy on analytical predictions of structural response. Often, in a preliminary analysis, anisotropy is neglected in the problem formulation, and the analysis of a composite structure is performed in a manner similar to the analysis for a structure made of an isotropic material. This simplifying assumption is valuable in that it permits the structural analyst to make use of existing solutions for specially orthotropic structures and to exploit symmetry in the structural modeling to reduce computational effort and cost. With the increasing use of composite materials for flight structures, the need has emerged to understand the effects and significance of anisotropy on structural response.

This paper addresses the consequences of neglecting anisotropy on the buckling of symmetrically laminated composite plates loaded in compression. During bending deformations, this class of composite plates displays anisotropy in the form of material-induced coupling between pure bending and twisting of the plate midplane due to the D_{16} and D_{26} constitutive terms. These anisotropic constitutive terms appear whenever a ply is stacked with a fiber orientation other than 0 or 90 deg to the reference axes of the plate. Hence, their values and importance depend on ply orientation, number of plies, and stacking sequence. Previous studies that address the importance of neglecting the D_{16} and D_{26} constitutive terms on the response of laminated plates are presented in Refs. 1-4. In Refs. 1 and 2, both experimental and analytical results obtained by the Galerkin method are presented for uniaxially compressed plates. These results indicate that significant nonconservative errors can occur in buckling predictions for some stacking sequences if the anisotropic constitutive terms in the analytical formulation are neglected. Furthermore, a criterion based on experimental results was proposed in Ref. 1 for determining when the anisotropic terms can be neglected. In Refs. 3 and 4, an energy measure was proposed to determine the influence of the anisotropic terms on the linear and nonlinear response of composite structures.

In this paper, a criterion is presented for assessing when the anisotropic constitutive terms can be neglected in the buckling analysis of uniaxially compressed laminates. The criterion is based on a parameter that is obtained from a nondimensionalization procedure similar to the procedure presented in Ref. 5. From the nondimensionalization procedure, nondimensional material coefficients are obtained that relate ply orientation, stacking sequence, and thickness effects to the anisotropic constitutive terms. Finite element analyses are used to verify the proposed criterion. The results presented in this paper are for balanced symmetrically laminated angle-ply and quasi-isotropic laminates.

Analysis

The differential equation governing buckling of symmetrically laminated composite plates is

$$D_{11}w,_{xxxx}+4D_{16}w,_{xxxy}+2(D_{12}+2D_{66})w,_{xxyy}+4D_{26}w,_{xyyy}$$
$$+D_{22}w,_{yyyy}=N_x w,_{xx}+N_y w,_{yy}+2N_{xy}w,_{xy} \qquad (1)$$

where N_x, N_y, and N_{xy} are membrane stress resultants prior to buckling. The symbols x and y are plate coordinates

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shown in Fig. 1, w is the out-of-plane deflection, D_{ij} are the plate bending stiffnesses, and commas followed by letters denote partial differentiation with respect to the coordinate corresponding to each letter. To arrive at the parameter representing the anisotropy of the problem, Eq. (1) is nondimensionalized following the procedure presented in Ref. 5. In this procedure, the equation is nondimensionalized to produce an equation without a preferred direction and to produce as few parameters as possible. In this paper, the x coordinate of the plate is nondimensionalized with respect to the half-wavelength of the buckling mode instead of the plate length. The resulting nondimensional form of Eq. (1) is

$$\begin{aligned}&\left[\frac{b}{\lambda}\left(\frac{D_{11}}{D_{22}}\right)^{1/4}\right]^2 w_{,\xi\xi\xi\xi} + 4\left[\frac{b}{\lambda}\left(\frac{D_{11}}{D_{22}}\right)^{1/4}\right]\\ &\times\left(\frac{D_{16}}{(D_{11}^3 D_{22})^{1/4}}\right) w_{,\xi\xi\xi\eta} + 2\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\right) w_{,\xi\xi\eta\eta}\\ &+4\left[\frac{\lambda}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}\right]\left(\frac{D_{26}}{(D_{11}D_{22}^3)^{1/4}}\right) w_{,\xi\eta\eta\eta}\\ &+\left[\frac{\lambda}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}\right]^2 w_{,\eta\eta\eta\eta} = \frac{N_x b^2}{\sqrt{D_{11}D_{22}}}\, w_{,\xi\xi}\\ &+\frac{N_y b^2}{D_{22}}\left[\frac{\lambda}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}\right]^2 w_{,\eta\eta}\\ &+2\left(\frac{N_{xy} b^2}{(D_{11}D_{22}^3)^{1/4}}\right)\left[\frac{\lambda}{b}\left(\frac{D_{22}}{D_{11}}\right)^{1/4}\right] w_{,\xi\eta}\end{aligned} \tag{2}$$

where ξ and η are the nondimensional coordinates given by

$$\xi = x/\lambda \qquad \eta = y/b \tag{3}$$

The constants a and b are the plate dimensions, and λ is the buckling mode half-wavelength shown in Fig. 1. The buckling results for any orthotropic plate loaded in uniaxial compression can be expressed in terms of a nondimensional buckling coefficient, written as

$$K = (N_x^{cr} b^2)/\pi^2\sqrt{D_{11}D_{22}} \tag{4}$$

a nondimensional wavelength parameter

$$\alpha = (b/\lambda)(D_{11}/D_{22})^{1/4} \tag{5}$$

and a twisting stiffness parameter

$$\beta = (D_{12}+2D_{66})/\sqrt{D_{11}D_{22}} \tag{6}$$

In this paper, two additional nondimensional material coefficients,

$$\gamma = D_{16}/(D_{11}^3 D_{22})^{1/4} \qquad \delta = D_{26}/(D_{11}D_{22}^3)^{1/4} \tag{7}$$

appear. These coefficients represent plate anisotropy and are referred to herein as the anisotropic coefficients.

Results and Discussion

The influence of anisotropic bending stiffnesses on buckling load depends on the relative importance of the odd mixed partial derivative terms appearing in Eq. (1). For some laminate stacking sequences, these terms may contribute significantly to the buckling load. For other laminate stacking sequences, the contribution of these terms to the buckling load may be negligible compared to the contribution of the other terms in Eq. (1). The nondimensionalization of the equation governing buckling is used in this paper to assess the contribution and importance of these terms.

In the nondimensionalization procedure, the values of the ξ and η coordinates have magnitudes of order one. Similarly, since Eq. (2) is homogeneous and the boundary conditions considered in this paper are homogeneous, the out-of-plane deflection can also be normalized to have magnitudes of order one. These two properties of Eq. (2) suggest that the derivatives appearing in Eq. (2) also have magnitudes of order one. From a more physical viewpoint, the buckling mode shapes for the laminates considered in this paper do not display any boundary-layer-like behavior that corresponds to derivatives having higher-order magnitudes. Using the notion that all the derivatives appearing in Eq. (2) have magnitudes of order one, it is assumed that the importance of anisotropy on the response is related to the relative numerical values of the nondimensional material coefficients (or combinations of them) multiplying the derivatives.

Nondimensional Material Coefficients

Values of the nondimensional material coefficients for the laminates considered in this study are shown in Figs. 2-4 as functions of fiber orientation, stacking sequence, and number of plies. The symbol m, referred to herein as the stacking sequence number, is used in this paper to represent the number of times a given arrangement of plies is repeated in a stacking sequence. All results presented in this paper are for laminates made of graphite-epoxy plies, having a longitudinal modulus $E_1 = 127.5$ GPa (18.5×10^6 psi), a transverse modulus $E_2 = 11.0$ GPa (1.6×10^6 psi), an in-plane shear modulus $G_{12} = 5.5$ GPa (0.8×10^6 psi), major Poisson's ratio $\nu_{12} = 0.35$, and a nominal ply thickness of 0.127 mm (0.005 in.). For the angle-ply laminates, the coefficients representing anisotropy (i.e., those with D_{16} and D_{26}) have maximum values for the 45-deg ply orientation (see Fig. 2) and their values diminish as the number of plies increase (see Fig. 3). The specially orthotropic coefficients (i.e., those with D_{11}, D_{12}, D_{22}, and D_{66}) have maximum values for the 0 and 45-deg ply orientations, and their values remain constant as the number of plies increase. For the 0/45/90 family of quasi-isotropic laminates, the coefficients representing anisotropy asymptotically approach zero from above as the basic laminates ply group of ($\pm$45/0/90) or (0/90/$\pm$45) is increasingly repeated. The solid lines and dashed lines shown in Fig. 4 represent the stacking sequences with the (0/90/$\pm$45) and ($\pm$45/0/90) basic ply groups, respectively. The remaining specially orthotropic coefficients asymptotically approach the value of one, from above or below, depending on the stacking sequence.

Finite Element Results

The finite element analyses were conducted using the computer program known as EAL.[6] A mesh refinement study

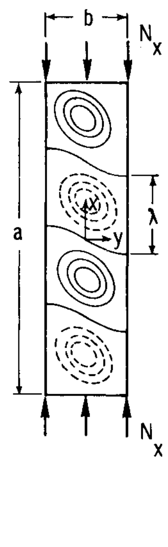

Fig. 1 Geometry of a buckled plate.

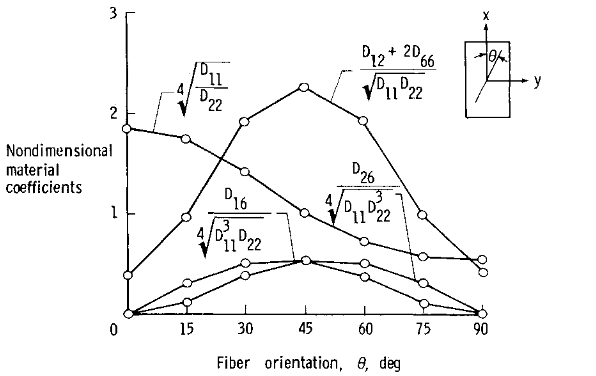

Fig. 2 Nondimensional material coefficients for $(\pm\theta)_s$ laminates.

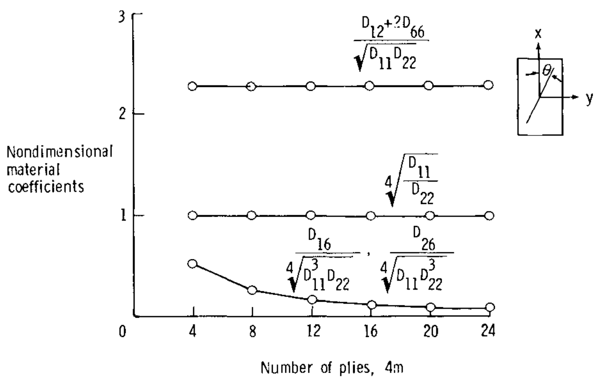

Fig. 3 Nondimensional material coefficients for $[(\pm 45)_m]_s$ laminates.

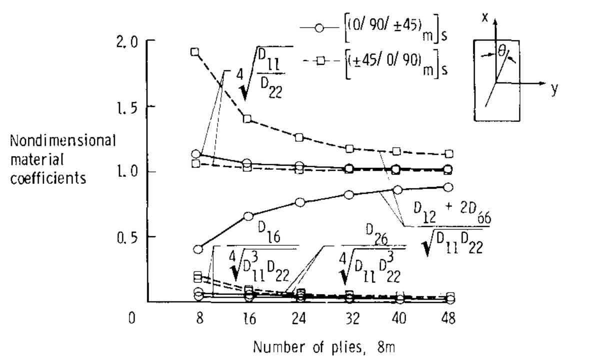

Fig. 4 Nondimensional material coefficients for quasi-isotropic laminates.

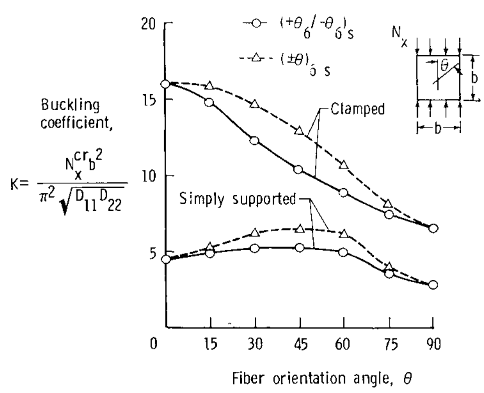

Fig. 5 Buckling coefficients for $[(\pm\theta)_6]_s$ and $(+\theta_6/-\theta_6)_s$ laminates.

was conducted to verify that the results presented in this paper are accurate.

The results shown in Figs. 5 and 6 indicate the effect of neglecting the D_{16} and D_{26} constitutive terms on the buckling of square 24-ply laminates that are either clamped or simply supported on all edges. The solid lines shown in Fig. 5 correspond to laminates constructed by stacking the plies consecutively with the same orientation in groups of six; i.e., $(+\theta_6/-\theta_6)_s$. The values of the nondimensional material coefficients for these laminates are the same as the values shown in Fig. 2 for the $(\pm\theta)_s$ laminates. Hence, these 24-ply laminates have the same degree of anisotropy as the corresponding angle-ply laminates with four alternating plies. The dashed lines shown in Fig. 5 correspond to 24-ply laminates in which the ply orientations alternate consecutively; i.e., $[(\pm\theta)_6]_s$. The specially orthotropic material coefficients for both stacking sequences shown in Fig. 5 are equal to the corresponding values given in Fig. 2. The values of the anisotropic material coefficients (see Fig. 3) for the $[(\pm\theta)_6]_s$ laminates are much smaller than the corresponding values for the $(+\theta_6/-\theta_6)_s$ laminates. Analysis conducted in this study indicates that the buckling loads obtained for the $[(\pm\theta)_6]_s$ laminates are within 1% of the buckling loads obtained by neglecting D_{16} and D_{26}. This result is illustrated in Fig. 6 for the angle-ply laminates that have 60-deg ply orientations. As the stacking sequence number m increases, the buckling coefficients converge monotonically from below to the specially orthotropic buckling coefficient. Contour plots of the buckling modes for the $[(\pm 60)_m]_s$ laminates (out-of-plane displacements and rotations) are shown in Fig. 7 and indicate that specially orthotropic deformation symmetry is present in those laminates with 24 or more plies. The laminates with less than 24 plies exhibit buckling mode shapes that are skewed from shapes with two perpendicular axes of deformation symmetry. The amount of the skew increases as the relative importance of the anisotropic coefficients increases. The results shown in Figs. 6 and 7 also indicate that buckling mode shape is more sensitive to the anisotropy than the buckling load. For instance, the results shown in Fig. 6 indicate that buckling loads obtained from analyses that include and neglect anisotropy, respectively, are nearly equal for the $[(\pm 60)_3]_s$ laminate. However, the results shown in Fig. 7 for the same laminate indicate that the buckling mode shape displays a substantially skewed shape unlike the corresponding specially orthotropic mode shape.

The results presented in Figs. 5 and 6 indicate small changes in the influence of the anisotropic terms due to changing boundary conditions and indicate substantial influence associated with changing the fiber orientation and stacking sequence. The largest effect of anisotropy, about 24% difference in the buckling loads, occurs at ply orientations of 45 deg (where the anisotropic material coefficients are maximum) for both the clamped and simply supported plates. Similar results for rectangular $(+60_6/-60_6)_s$ and $[(\pm 60)_6]_s$ laminates are shown in Fig. 8. The aspect ratios considered range from 1 to 4. The results shown in Fig. 8 indicate that aspect ratio does not affect the influence of anisotropy. For each of these laminates, the aspect ratio of the buckles occurring in the plate is approximately equal to 1 (nearly square).

Buckling results for the quasi-isotropic laminates are shown in Fig. 9. These results indicate that the buckling coefficients converge asymptotically to the corresponding isotropic solution as the number of plies increase. Also, depending on the location of the 45-deg plies, the isotropic buckling solution may be a conservative or nonconservative estimate of the actual buckling solution. The effect of neglecting the anisotropic constitutive terms in the buckling analysis is negligible for both quasi-isotropic stacking sequences considered, with the exception of the $(\pm 45/0/90)_s$ laminate. Neglecting anisotropy in the buckling analysis for

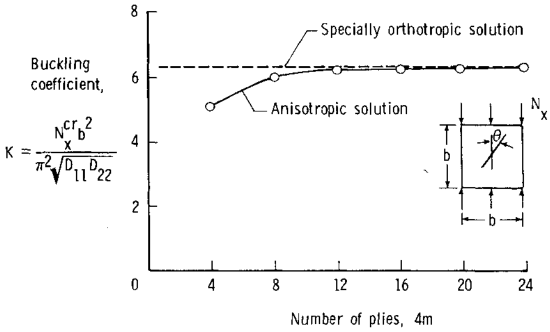

Fig. 6 Buckling coefficients for simply supported $[(\pm 60)_m]_s$ laminates.

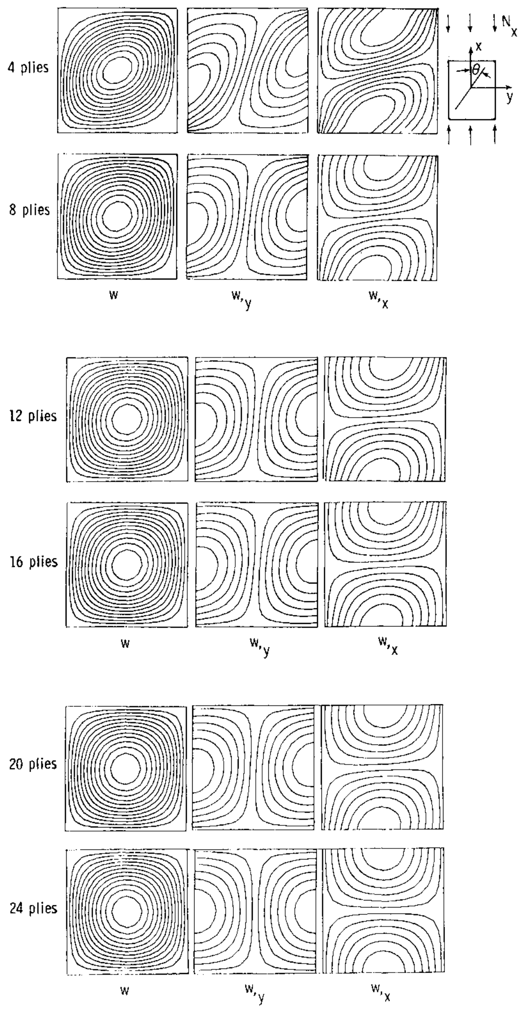

Fig. 7 Buckling modes for simply supported square $[(\pm 60)_m]_s$ laminates.

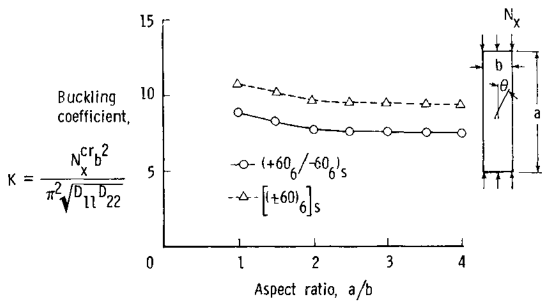

Fig. 8 Buckling coefficients for clamped rectangular $(+60_6/-60_6)_s$ and $[(\pm 60)_6]_s$ laminates.

this laminate results in a buckling load 3% higher than the buckling load including anisotropy.

Criterion for Assessing the Importance of Anisotropy

Using the notion that the derivatives appearing in Eq. (2) have magnitudes of order one, it appears that the importance of neglecting anisotropy on the buckling predictions is related to the relative influence of the nondimensional material coefficients multiplying these derivatives. Based on this idea, the anisotropy is assumed not to be negligible when the anisotropic coefficients given by Eq. (7) are of the same order as the orthotropic coefficients given by Eqs. (5) and (6). To determine the relative orders of these coefficients, and hence the importance of the anisotropy, it is useful to express the left-hand side of Eq. (2) as

$$\alpha^2 w,_{\xi\xi\xi\xi} + 4\alpha\gamma w,_{\xi\xi\xi\eta} + 2\beta w,_{\xi\xi\eta\eta} + 4\delta/\alpha w,_{\xi\eta\eta\eta} + 1/\alpha^2 w,_{\eta\eta\eta\eta} \tag{8}$$

The results shown in Fig. 2 indicate that for the laminates considered in this paper, the twisting stiffness parameter β has the range

$$0.4 < \beta < 2.3 \tag{9}$$

For the orthotropic coefficients appearing in Eq. (8) to be of the same order of importance, the coefficients must satisfy the conditions

$$\alpha^2 = \mathcal{O}(2\beta) \tag{10}$$

and

$$1/\alpha^2 = \mathcal{O}(2\beta) \tag{11}$$

Examining these two conditions with respect to the range of β given by Eq. (9) suggests that the range of α is

$$0.9 < \alpha < 1.1 \tag{12}$$

This result implies that the effective aspect ratio of the buckles occurring in the plate is near the value of 1 (nearly square).

For the anisotropic coefficients to be of the same order as the orthotropic coefficients, the coefficients in Eq. (8) must satisfy the conditions

$$\gamma = \mathcal{O}\begin{Bmatrix} \alpha/4 \\ 1/4\alpha^3 \\ \beta/2\alpha \end{Bmatrix} \tag{13}$$

and

$$\delta = \mathcal{O}\begin{Bmatrix} \alpha^3/4 \\ 1/4\alpha \\ \beta\alpha/2 \end{Bmatrix} \tag{14}$$

Satisfaction of these conditions leads to the range of the anisotropic coefficients being

$$0.2 < \gamma, \delta < 1.3 \tag{15}$$

When the maximum value of γ and δ is less than 0.2, the anisotropy is expected to be negligible.

The results shown in Fig. 10 indicate the percent change in the buckling coefficient obtained by neglecting the anisotropic constitutive terms as a function of an anisotropic parameter defined by

$$\mu = \text{maximum}(\gamma, \delta) \tag{16}$$

In Fig. 10, K^* is the value of the buckling coefficient K [see Eq. (4)] obtained when D_{16} and D_{26} are neglected in the

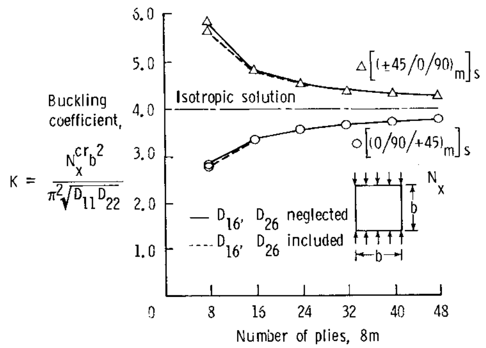

Fig. 9 Buckling coefficients for simply supported quasi-isotropic laminates.

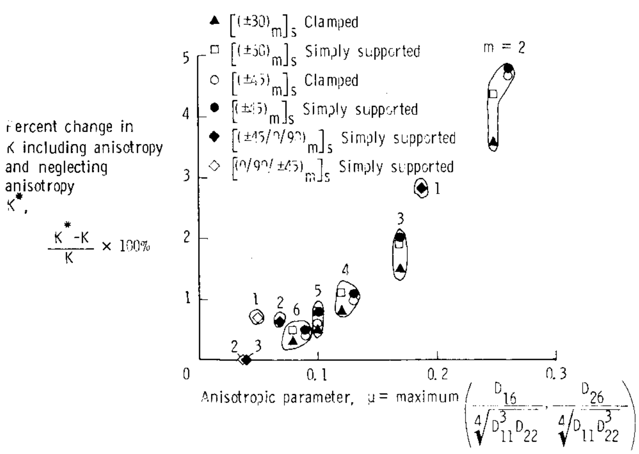

Fig. 10 Percent change in the buckling coefficient due to neglecting anisotropy.

analysis. Results are shown for clamped and simply supported angle-ply laminates and simply supported quasi-isotropic laminates, and apply to both square and rectangular geometries (based on the results shown in Fig. 8). The results presented in Fig. 10 indicate that the laminates having values of the anisotropic parameter μ less than 0.18 will have buckling loads that are at most 2% higher than the corresponding specially orthotropic solutions. For laminates with $\mu \geq 0.18$, neglecting the anisotropy results in nonconservative errors in the buckling load ranging from slightly more than 2% to as much as 25%. The larger error corresponds to results for the $(\pm 30)_s$, $(\pm 45)_s$, and $(\pm 60)_s$ laminates, not shown in Fig. 10. The results for the quasi-isotropic laminates indicate that the importance of the anisotropy is greater in the laminates with the 45-deg plies located at the top and bottom surfaces, especially in the laminates with eight plies.

Concluding Remarks

The results of a study of the effects of anisotropy on the buckling loads of composite plates loaded in compression have been presented. Using a nondimensionalization procedure, nondimensional material coefficients of the differential equation governing buckling are obtained and their relationship to the D_{16} and D_{26} anisotropic constitutive terms is presented on a rational basis. In particular, it is shown that the relative numerical values of the nondimensional material coefficients containing the anisotropic constitutive terms, compared to the remaining nondimensional material coefficients, diminish as the number of alternating plies making up the laminate increases. The largest values of the anisotropic material coefficients are shown to occur at 45-deg ply orientations.

Using accurate finite element analyses, it is shown that the 45-deg angle-ply laminates exhibit the strongest influence of anisotropy on the buckling load. Similarly, it is shown that the influence of anisotropy is independent of aspect ratio and the clamped and simply supported boundary conditions considered. Furthermore, the influence of anisotropy on the quasi-isotropic laminates considered is shown to be negligible, with the exception of the $(\pm 45/0/90)_s$ laminate, which exhibits a buckling load 3% less than the corresponding buckling load obtained from an analysis that neglects anisotropy. The importance of the anisotropy is shown to be typically less important in the $[(0/90/\pm 45)_m]_s$ laminates than in the $[(\pm 45/0/90)_m]_s$ laminates (m refers to the number of times the basic arrangement of plies is repeated). The buckling loads for the quasi-isotropic laminates are shown to converge to the corresponding isotropic buckling load as the number of plies making up the laminate increases.

Based on the results of the finite element analyses presented and the nondimensionalization procedure, a nondimensional parameter is presented that is used to ascertain when anisotropy is negligible in a buckling analysis. For laminates having values of the nondimensional parameter less than 0.18, nonconservative errors in the buckling predictions obtained from a specially orthotropic analysis of less than 2% can be expected. For laminates with values of the parameter ≥ 0.18, nonconservative errors in the buckling load as high as 25% may occur.

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